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A note on solving the fuzzy Sylvester matrix equation

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Abstract

In this work, we present theoretical analysis of the solution of Fuzzy Sylvester Matrix Equation (FSME) in the form $A\tilde{X} + \tilde{X}B = \tilde{C}$. The necessary and sufficient conditions for the existence of fuzzy solutions are proposed and some operators to finding the solution of FSME are exploited. Furthermore, an iterative scheme which can solve two $n \times n$ system instead one $2n \times 2n$ system is presented to solving extended fuzzy linear system. Several numerical examples are performed to illustrate developed theory.

keywords: Fuzzy Sylvester matrix equation, Fuzzy linear system, Iterative method.

2000 AMS Subject Classification: 15A06

1 Introduction

The matrix equation

$$AX + XB = C, \qquad (1.1)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{m \times m}$ and $C \in \mathbb{R}^{n \times m}$ are known matrices and $X \in \mathbb{R}^{n \times m}$ is an unknown matrix is called a Sylvester matrix equation. This equation is of considerable importance in many applications such as system theory, control theory, matrix problems arising in partial differential equations and some techniques in ordinary deferential equations [8], [9], [14]. The solution of the Sylvester matrix equations uses two types of methods. The first type of methods is based on the transforming the coefficient matrices to Schur or Hessenberg form and then solving corresponding linear system directly. The

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second type of method are iterative scheme methods focusing on large sparse systems [8], [9], [14]. Now, consider the matrix equation

$$A\tilde{X} + \tilde{X}B = \tilde{C}; \tag{1.2}$$

where $\tilde{X} \in \mathbb{R}^{n\times m}$ is an unknown fuzzy matrix and $\tilde{C} \in \mathbb{R}^{n\times m}$ is a known fuzzy matrix. This paper will investigate the solution of (1.2) which is called a "fuzzy Sylvester matrix equation".

It is known that the systems of linear equations play major role in various areas such as science, engineering and economics. The concept of fuzzy numbers and arithmetic operations associated with these numbers was first presented by Zadeh [20]. Friedman et al. in [12] proposed a model for solving an $n \times n$ fuzzy linear system of equation where the coefficient matrix is crisp and the right-hand side column is an arbitrary vector of fuzzy number. They used the embedding method and replaced the original $n \times n$ fuzzy linear system by $2n \times 2n$ crisp function linear system. Authors such as Abbasbandy et al. in [1], [2], Allahviranloo et al. in [3], [4] and Dehghan and Hashemi in [10] have extended the work of Friedman.

Recently, some authors focus on solving fuzzy matrix equations. Zengtai and Xiaobin in [13] investigated the fuzzy matrix equation in the form of $A\tilde{X} = \tilde{B}$. They used generalized matrix inverse and presented the least square solution to the fuzzy matrix equation. Allahviranloo et al. in [5] presented a two stage method for computing fuzzy linear matrix equation $A\tilde{X}B = \tilde{C}$. They applied the Kronecker product to transform this fuzzy linear matrix equation to a non-square system and used the embedding approach. Salkuyeh [18] investigated the fuzzy Sylvester matrix equation where the crisp matrices A and B are M-matrices. He has solved the associated system by accelerated over relaxation method.

In this paper we investigate analytically and practically the fuzzy Sylvester matrix equation $A\tilde{X} + \tilde{X}B = \tilde{C}$ where A is an $n \times n$ matrix, B is a $m \times m$ matrix and $\tilde{C}$ is an $m \times n$ fuzzy matrix. Some operators such as Kronecker product, Kronecker sum and vec-operator are exploited to transform the fuzzy Sylvester matrix equation to $mn \times mn$ fuzzy linear system. An iterative method which has feasible property will be introduced for solving the associated fuzzy linear system. Furthermore, an interesting application of fuzzy Sylvester matrix equation is introduced. Throughout this note, the Kronecker product of two matrices denoted by $A \otimes B$, the vector of each matrix $A \in \mathbb{C}^{m\times n}$ presented by $\mathrm{vec}(A) \in \mathbb{C}^{mn}$, and the spectrum of an arbitrary matrix W introduced by $\sigma(W)$. The outline of the paper is as follows: In Section 2 we will present basic definitions and concepts. In Section 3 we will discuss about the necessary and sufficient condition for existence of solution of FSME. In section 4, an algorithm will be presented for solving fuzzy linear system. Numerical examples will be given in Section 5 and the conclusions are in Section 6.

2 Basic concepts

In this section, some brief preliminary concepts of the fuzzy number arithmetic and fuzzy Sylvester matrix equation are given.

Definition 2.1: [10] A fuzzy number is defined by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$, which satisfy the following conditions:

1. $\underline{u}(r)$ is a bounded left continuous non-decreasing function over $[0, 1]$;
2. $\overline{u}(r)$ is a bounded left continuous non-increasing function over $[0, 1]$;
3. $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

A crisp number α is represented by $\underline{u}(r) = \overline{u}(r) = \alpha$, $0 \leq r \leq 1$. For an arbitrary fuzzy number $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ and $\lambda \in \mathbb{R}$, arithmetic operators can be defined as [10]:

$$\begin{aligned}
&u = v \quad \text{iff} \quad \underline{u}(r) = \underline{v}(r) \quad \text{and} \quad \overline{u}(r) = \overline{v}(r), \\
&u + v = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)) \\
&u - v = (\underline{u}(r) - \overline{v}(r), \overline{u}(r) - \underline{v}(r)) \\
&\lambda u = \begin{cases} (\lambda \underline{u}(r), \lambda \overline{u}(r)), & \lambda \geq 0, \\ (\lambda \overline{u}(r), \lambda \underline{u}(r)), & \lambda < 0. \end{cases}
\end{aligned} \tag{2.1}$$

Definition 2.2: The Sylvester matrix equation $A\tilde{X} + \tilde{X}B = \tilde{C}$ is called a fuzzy Sylvester equation (FSME) if matrices $A = (a_{ij}), 1 \leq i, j \leq n$ and $B = (b_{ij}), 1 \leq i, j \leq m$ are crisp matrices and right-hand side matrix $\tilde{C} = (c_{ij}), 1 \leq i \leq n, 1 \leq j \leq m$ is a known fuzzy matrix, where $\tilde{X} = (x_{ij}), 1 \leq i \leq n, 1 \leq j \leq m$ is an unknown fuzzy matrix. Note that the ijth equation of this system is:

$$\sum_k a_{ik}x_{kj} + \sum_\ell x_{i\ell}b_{\ell j} = c_{ij}, \qquad 1 \leq i \leq n, 1 \leq j \leq m. \tag{2.2}$$

By writing the matrices in (1.2) in terms of their columns, it is easily seen by equating ith column we have the relation

$$A\mathbf{x}_i + \tilde{X}\mathbf{b}_i = A\mathbf{x} + \sum_{j=1}^{m} b_{ji}x_j = \mathbf{c}_i \tag{2.3}$$

These equations can be written as the $mn \times mn$ fuzzy linear system as follows:

$$\begin{pmatrix} A + b_{11}I & b_{21}I & \cdots & b_{m1}I \\ b_{12}I & A + b_{22}I & \cdots & b_{m2}I \\ \vdots & \vdots & \ddots & \vdots \\ b_{1m}I & b_{2m}I & \cdots & A + b_{mm}I \end{pmatrix} \begin{pmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ \vdots \\ \mathrm{x}_m \end{pmatrix} = \begin{pmatrix} \mathrm{c}_1 \\ \mathrm{c}_2 \\ \vdots \\ \mathrm{c}_m \end{pmatrix}. \tag{2.4}$$

Clearly, the system of equation (2.4) can be replaced in the form

$$(A \oplus B^t)\mathrm{vec}(\tilde{X}) = \mathrm{vec}(\tilde{C}); \tag{2.5}$$

where $(A \oplus B^t)$ is Kronecker sum of matrices A and B that can be defined as $(A \oplus B^t) = (I_m \otimes A) + (B^t \otimes I_n)$. In continue of this paper, we use $\mathrm{vec}(W) = \mathrm{w}$ (small and bold) for given matrix $W \in \mathbb{C}^{m \times n}$. Now, in the following definition we define a solution of the FSME.

Definition 2.3: A fuzzy number matrix $\tilde{X} = (x_{ij}); \; 1 \le i \le n; 1 \le j \le m$ given by $x_{ij} = (\underline{x}_{ij}(r); \overline{x}_{ij}(r)); \; 1 \le i \le n; 1 \le j \le m$ is called a solution of the FSEM if:

$$\begin{aligned} \underline{\sum_k a_{ik}x_{kj} + \sum_\ell x_{i\ell}b_{\ell j}}(r) &= \underline{\sum_k a_{ik}x_{kj}}(r) + \underline{\sum_\ell x_{i\ell}b_{\ell j}}(r) = \underline{c_{ij}}(r); \\ \overline{\sum_k a_{ik}x_{kj} + \sum_\ell x_{i\ell}b_{\ell j}}(r) &= \overline{\sum_k a_{ik}x_{kj}}(r) + \overline{\sum_\ell x_{i\ell}b_{\ell j}}(r) = \overline{c_{ij}}(r). \end{aligned} \tag{2.6}$$

From (3.1) in particular , if $a_{ij} \ge 0$, for $1 \le i \le n; 1 \le j \le m$ and $b_{ij} \ge 0$, for $1 \le i \le n; 1 \le j \le m$ or if $a_{ii} + b_{jj} \ge 0$, for $1 \le i \le n; 1 \le j \le m$, we simply get:

$$\begin{aligned} \underline{\sum_k a_{ik}x_{kj} + \sum_\ell x_{i\ell}b_{\ell j}}(r) &= \sum_k a_{ik}\underline{x_{kj}}(r) + \sum_\ell \underline{x_{i\ell}}(r)b_{\ell j} = \underline{c_{ij}}(r); \\ \overline{\sum_k a_{ik}x_{kj} + \sum_\ell x_{i\ell}b_{\ell j}}(r) &= \sum_k a_{ik}\overline{x_{kj}}(r) + \sum_\ell \overline{x_{i\ell}}(r)b_{\ell j} = \overline{c_{ij}}(r). \end{aligned} \tag{2.7}$$

In general, an arbitrary equation may include a linear combination of $\underline{x}_{ij}$'s and $\overline{x}_{ij}$'s for either $\underline{c}_{ij}$ or $\overline{c}_{ij}$. Hence, where the right hand side vector is the function vector $C = \begin{pmatrix} \underline{\mathbf{c}} \\ \overline{\mathbf{c}} \end{pmatrix}$, one can solve a crisp linear system in order to solve the system given by (2.6) and (2.7).

3 Fuzzy Sylvester Matrix Equation

In this section we discuss some aspects related to the FSME. First, in the following theorems we are ready to state the condition which the FSME must satisfy to have a unique fuzzy solution. For this purpose, we need to find the eigenvalues and eigenvectors of $A \oplus B$.

Theorem 3.1: [16] Let $A \in \mathbb{R}^{n \times n}$ have eigenvalues $\lambda_i; i = 1, \ldots, n$ and $B \in \mathbb{R}^{n \times n}$ have eigenvalues $\mu_j; j = 1, \ldots, m$. Then the kronecker sum $A \oplus B$ have mn eigenvalues

$$\lambda_1 + \mu_1, \ldots, \lambda_1 + \mu_m, \lambda_2 + \mu_1, \ldots, \lambda_2 + \mu_m, \ldots, \lambda_n + \mu_1, \ldots, \lambda_n + \mu_m.$$

Moreover, if $x_1, \ldots, x_p$ are linearly independent right eigenvectors of A corresponding to $\lambda_1, \ldots, \lambda_p (p \leq n)$, and $z_1, \ldots, z_p$ are linearly independent right eigenvectors of B corresponding to $\mu_1, \ldots, \mu_q (q \leq m)$, then $z_j \otimes x_i \in \mathbb{R}^{mn}$ are linearly independent right eigenvectors of $A \oplus B$ corresponding to $\lambda_i + \mu_j; \quad i = 1, \ldots, p; \quad j = 1, \ldots, q$.

We can now prove the following theorem.

Theorem 3.2: Suppose that $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{m \times m}$ are crisp matrices, and $\tilde{C}$ is $m \times n$ fuzzy matrix. Then the fuzzy Sylvester matrix equation $A\tilde{X} + \tilde{X}B = \tilde{C}$ has a unique fuzzy solution if and only if A and $-B$ have no eigenvalue in common. In other words, $\sigma(A) \cap \sigma(-B) = \emptyset$.

proof: According to definition of Kronecker sum, the equation (1.2) can be written as follows:

$$(A \oplus B^t)\tilde{x} = \tilde{c};$$

Suppose that FSME has a unique fuzzy solution. The equation $A\tilde{X} + \tilde{X}B = \tilde{C}$ has a unique fuzzy solution if and only if $(A \oplus B^t)$ is a nonsingular matrix. But we know that $(A \oplus B^t)$ is a nonsingular matrix if and only if it has non zero eigenvalues. According to Theorem 3.1 we then have

$$(A \oplus B^t)(\lambda_i + \mu_j) = (\lambda_i + \mu_j)(\tilde{x} \otimes \tilde{z})$$

where $\lambda_i \in \sigma(A); i = 1, \ldots, n$ and $\mu_j \in \sigma(B); j = 1, \ldots, m$. Hence we conclude $\sigma(A) \cap \sigma(-B) = \emptyset$.

Conversely, suppose that $\sigma(A) \cap \sigma(-B) = \emptyset$. We would like to show that the equation $A\tilde{X} + \tilde{X}B = \tilde{C}$ has a unique solution. Let us put $T = \begin{pmatrix} A & \tilde{C} \\ 0 & -B \end{pmatrix}$ and $Z = \begin{pmatrix} I_m & \tilde{X} \\ 0 & I_n \end{pmatrix}$. It is clear that

$$Z^{-1}TZ = \begin{pmatrix} I_m & \tilde{X} \\ 0 & I_n \end{pmatrix}^{-1} \begin{pmatrix} A & \tilde{C} \\ 0 & -B \end{pmatrix} \begin{pmatrix} I_m & \tilde{X} \\ B & I_n \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix}$$

Thus the two matrices $\begin{pmatrix} A & \tilde{C} \\ 0 & -B \end{pmatrix}$ and $\begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix}$ are similar and have the same set of eigenvalues. According to [15], this condition is equivalent to the condition for existence of a unique solution of $A\tilde{X} + \tilde{X}B = \tilde{C}$.

In this part, we are going to rearrange the FSME so that the unknowns are $\underline{x}_{ij}$, $-\overline{x}_{ij}$, for $1 \le i \le n, 1 \le j \le m$. For this purpose, first we compute the coefficient matrix $Q = (A \oplus B^t)$. Clearly, we can state that

$$Q = (q_{i,j}) \in \mathbb{R}^{mn \times mn}, \quad i,j = 1,2,\dots,mn$$

Where, the $2mn \times 2mn$ matrix $S = (s_{ij})$ is determined as follows:

If $q_{ij} \ge 0$ for $i,j = 1,\dots,mn$, then we can put $s_{i,j} = q_{ij}$ and $s_{i+n,j} = q_{ij}$.

If $q_{ij} < 0$ for $i,j = 1,\dots,mn$, then we can put $s_{i,j+n} = -q_{ij}$ and $s_{i+n,j} = -q_{ij}$.

Thus, the system $A\tilde{X} + \tilde{X}B = \tilde{C}$ is extended to the following crisp linear system

$$SX = C \tag{3.1}$$

where $S = (s_{ij})$ for $1 \le i,j \le 2mn$ and $X = \begin{pmatrix} \underline{x} & -\overline{x} \end{pmatrix}^t, C = \begin{pmatrix} \underline{c} & -\overline{c} \end{pmatrix}^t.$ The structure of S implies that $s_{ij} \ge 0$ for $1 \le i,j \le 2mn$ and

$$S = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix} \tag{3.2}$$

where S_1 contains the positive entries of $(A \oplus B^t)$, S_2 contains the absolute values of the negative entries of $(A \oplus B^t)$. Thus we have $A \oplus B^t = S_1 - S_2$, which implies that the rest of the entries are zero. As can be seen, the Sylvester matrix equation is a general crisp system of linear equations and we can solve $2mn \times 2mn$ system for X. But, we should answer this important question: “Does a crisp function linear system have a solution when the FSME has a solution?" The next example shows that it is possible for $SX = C$ to have no solution or an infinite number of solutions even if $A\tilde{X} + \tilde{X}B = \tilde{C}$ has a unique solution.

Example 3.1: Consider the fuzzy Sylvester matrix equation $A\tilde{X} + \tilde{X}B = \tilde{C}$ where

$$A = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} (-1+r, 1-r) & (-2+3r, 4-3r) \\ (-5+6r, 2-r) & (-2+2r, 5-5r) \end{pmatrix}.$$

The coefficient of the left hand side system

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix},$$

is a nonsingular matrix and hence $(A \oplus B^t)x = c$ has a unique solution, while $S = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}$ is singular, because S_1 and S_2 are singular matrices. It can be seen that a FSME may have no solution or an infinite number of solution.

If A_+ and B_+ contain the positive entries of A and B respectively and A_- and B_- contain the negative entries of A and B respectively, it is obvious that $A = A_+ - A_-$ and $B = B_+ - B_-$. So, by simplifying the expression $(I_m \otimes A) + (B^t \otimes I_n) = S_1 - S_2$, then S_1 and S_2 can be obtained as

$$S_1 = (A_+ \otimes B_+^t); \quad S_2 = (A_- \otimes B_-^t); \tag{3.3}$$

Thus, it can be concluded

$$S_1 + S_2 = (A_+ + A_-) \otimes (B_+ + B_-)^t. \tag{3.4}$$

Theorem 3.3: [12] The matrix S is nonsingular if and if only the matrices $S_1 - S_2$ and $S_1 + S_2$ are both nonsingular.

Corollary 3.1: Suppose that A, B are square matrices. Then $S = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}$ is nonsingular matrix if and only if matrices $(A_- \otimes B^t)$, $(A_+ + A_-) \otimes (B_+ + B_-)^t$ are nonsingular.

Corollary 3.2: Let matrix S be in the form introduced $S = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}$. Then we have

$$S^{-1} = \frac{1}{2}\begin{pmatrix} ((A_+ + A_-) \otimes (B_+ + B_-)^t)^{-1} + (A_- \otimes B^t)^{-1} & ((A_+ + A_-) \otimes (B_+ + B_-)^t)^{-1} - (A_- \otimes B^t)^{-1} \\ ((A_+ + A_-) \otimes (B_+ + B_-)^t)^{-1} - (A_- \otimes B^t)^{-1} & ((A_+ + A_-) \otimes (B_+ + B_-)^t)^{-1} + (A_- \otimes B^t)^{-1} \end{pmatrix} \tag{3.5}$$

Corollary 3.3: If a crisp linear system does not have a solution, the associated fuzzy Sylvester linear system does not have one too. In order to solve the fuzzy linear system $SX = C$ directly, S^{-1} must be calculated (whenever it exists).

The next theorem provides information related to the structure of S^{-1}.

Theorem 3.4: [12] The unique solution X of Equation (3.4) is a fuzzy vector for arbitrary C if and only if S^{-1} is nonnegative. i.e, $(s_{ij})^{-1} \geq 0$.

By using result of Theorem 3.4 and corollary 3.2, we are ready to present necessary and sufficient condition for the existence of the solution to the system $SX = C$.

Theorem 3.5: The necessary and sufficient conditions for $SX = C$ to have a solution is that $((A_+ + A_-) \otimes (B_+ + B_-)^t)^{-1}\mathrm{x} = \mathrm{u}$ and $(A_- \otimes B^t)\mathrm{x} = \mathrm{v}$ should have a solution, where $\mathrm{u} = \underline{\mathrm{c}} - \overline{\mathrm{c}}$ and $\mathrm{v} = \underline{\mathrm{c}} + \overline{\mathrm{c}}$.

proof: According to [17] a necessary and sufficient condition for non-square system $SX = C$ to be consistent is $SS^{-}C = C$ where S^{-} is a generalized inverse of matrix S. If we consider square system, then it can be written $SS^{-1}C = C$. Now $SS^{-1}C = C$ if and only if:

$$\begin{cases} \frac{1}{2}[((A_+ + A_-) - (B_+ + B_-)^t)((A_+ + A_-) - (B_+ + B_-)^t)^{-1}(\underline{c} + \overline{c}) \\ \qquad + (A_- - B_-^t)^{-1}(A_- - B_-^t)(\underline{c} + \overline{c})] = \underline{c}; \\ \frac{1}{2}[((A_+ + A_-) - (B_+ + B_-)^t)((A_+ + A_-) - (B_+ + B_-)^t)^{-1}(\underline{c} - \overline{c}) \\ \qquad + (A_- - B_-^t)^{-1}(A_- - B_-^t)(\underline{c} - \overline{c})] = -\overline{c}; \end{cases}$$

i.e., if and only if

$$\begin{cases} ((A_+ + A_-) - (B_+ + B_-)^t)((A_+ + A_-) - (B_+ + B_-)^t)^{-1}(\underline{c} - \overline{c}) = \underline{c} - \overline{c}; \\ (A_- - B_-^t)(A_- - A_-^t)^{-1}(\underline{c} + \overline{c}) = (\underline{c} + \overline{c}); \end{cases}$$

i.e., if and only if

$$\begin{cases} ((A_+ + A_-) - (B_+ + B_-)^t)\mathrm{x} = \mathrm{u} \\ (A_- - B_-^t)\mathrm{x} = \mathrm{v} \end{cases}$$

where $\mathrm{u} = \underline{c} - \overline{c}$ and $\mathrm{v} = \underline{c} + \overline{c}$ are consistent.

Since $\underline{c}$ and $\overline{c}$ are linear combinations of $\underline{c}_{ij}$ and $\overline{c}_{ij}$, then $\underline{c}$ and $\overline{c}$ are bounded and left continuous. Hence, we should have the following properties for sufficient condition for one of solution of (3.1) to be fuzzy solution vector of (2.5) [5]:

1. $\underline{c}(r)$ monotonically increasing.
2. $\overline{c}(r)$ monotonically decreasing.
3. For $0 \leq r \leq 1$ we have $\underline{c}(r) \leq \overline{c}(r)$.

Now, we are ready to define fuzzy solution to the linear system.

Remark 3.6: More recently Allahviranloo et al. in [6] has proposed that the fuzzy solution of the fuzzy linear systems, introduced by Friedman et al. [12] may not be a fuzzy numbers vector. In other words, in that case it may be at least one of vector's components is not fuzzy number. It should be emphasized that in this work the authors considered the fuzzy solutions which are fuzzy vector numbers.

4 New iteration for solving fuzzy linear systems

System (3.1) can be solved directly by using of the inverse of S. In this section, an iteration to solve fuzzy linear system (3.1) is presented.

Suppose that $A \in \mathbb{R}^{n\times n}$ be a full-rank matrix to be determined, $Ax = y$ is the linear system and μ be the step-size or convergence factor. Ding and Chen in [11] proposed a family of iterative methods for solving linear systems which $\lim_{k\to\infty} x^{(k)} = x$ as follows:

$$x^{(k+1)} = x^{(k)} + \mu G[y - Ax^{(k)}]; \quad 0 < \mu < \frac{2}{\|A\|^2} \tag{4.1}$$

Now suppose that we take $G = S^T$, $A = S$, $X = \begin{pmatrix}\underline{x}\\ \overline{x}\end{pmatrix}$, and $C = \begin{pmatrix}\underline{c}\\ \overline{c}\end{pmatrix}$ then the following recursion converges to X:

$$X^{(k+1)} = X^{(k)} + \mu S^T[C - SX^{(k)}]; \quad 0 < \mu < \frac{2}{\|S\|^2} \tag{4.2}$$

Thus, the following algorithm is given:

```
Algorithm 1: (Iterative method for solving Eq.(3.1))
1. Choose initial guesses $\underline{x}^{(0)}$ and $\overline{x}^{(0)}$;
2. Compute $S_1^T$ and $S_2^T$;
3. For $k = 0; 1; 2; \ldots$ Do
4. $\underline{x}^{(k+1)} = \underline{x}^{(k)} + \mu S_1^T(\underline{c} - S_1\underline{x}^{(k)} + S_2\overline{x}^{(k)}) + S_2^T(-\overline{c} - S_2\underline{x}^{(k)} + S_1\overline{x}^{(k)})$ ;
5. $\overline{x}^{(k+1)} = \overline{x}^{(k)} + \mu S_2^T(-\underline{c} + S_1\underline{x}^{(k)} - S_2\overline{x}^{(k)}) + S_1^T(\overline{c} + S_2\underline{x}^{(k)} - S_1\overline{x}^{(k)})$ ;
6. If the stopping criterion satisfied, then stop,
7. End do.
```

We note that S_1^T and S_2^T are transpose of matrices S_1 and S_2.

5 Numerical examples

In this section, some numerical experiments are given to illustrate the presented theoretical results. The computations are carried out by MATLAB 7.10. In addition, the triangular fuzzy numbers are used in all of the following numerical examples. For a triangular fuzzy number $u = (\underline{u}(r); \overline{u}(r)) = (a + br; c + dr)$, its norm as in Wang and Zheng [19] is defined as $\|x\| = \max\{|a|; |b|; |c|; |d|\}$. It must be noticed that for stopping, the norms of

$$\frac{\underline{x}^{(k+1)} - \underline{x}^{(k)}}{\underline{x}^{(k+1)}} < \varepsilon \quad \text{and} \quad \frac{\overline{x}^{(k+1)} - \overline{x}^{(k)}}{\overline{x}^{(k+1)}} < \varepsilon$$

can be used where $\varepsilon = 10^{-4}$.

Example 4.1: For the first example consider FSME, $A\tilde{X} + \tilde{X}B = \tilde{C}$, whenever

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}; \quad B = \begin{pmatrix} 2 & 3 \\ 0 & 3 \end{pmatrix};$$

and fuzzy right hand side

$$\tilde{C} = \begin{pmatrix} (15+17r, 19-17r) & (26+26r, 30-30r) \\ (24+31r, 38-31r) & (42+47r, 56-51r) \end{pmatrix}.$$

The exact solution of this system is

$$\tilde{X} = \begin{pmatrix} (3+4r, 5-4r) & (1+2r, 4-3r) \\ (4+5r, 6-5r) & (5+6r, 7-6r) \end{pmatrix}$$

Clearly, $(A) = \{1.0000 + 2.0000i, 1.0000 - 2.0000i\}$ and $(-B) = \{-2, -3\}$. Since $(A) \cap (-B) = \emptyset$, the condition of the Theorem 3.3 is satisfied. The coefficient and right hand side of the system $(A \oplus B^t)\mathbf{x} = \mathbf{c}$ are as following

$$Q = \begin{pmatrix} 3 & -1 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ -3 & 0 & 4 & -1 \\ 0 & -3 & 4 & 4 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} (15+17r, 19-17r) \\ (24+31r, 38-31r) \\ (26+26r, 30-30r) \\ (42+47r, 56-51r) \end{pmatrix},$$

Therefore, the coefficient of the extended 8×8 system $SX = C$ can be written as $S = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}$ where

$$S_1 = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 4 & 4 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 \end{pmatrix}$$

The extended system $SX = C$ can be solved by utilizing Algorithm 1. By taking $\;= 0.0308$ after 536 iteration we obtain

$$\tilde{X} = \begin{pmatrix} (2.9997+3.9997r, 4.9997-3.9997r) & (1.0005+2.0005r, 4.0005-3.0005r) \\ (4.0005+5.0005r, 6.0005-5.0005r) & (4.9990+5.9990r, 6.9990-5.9990r) \end{pmatrix}.$$

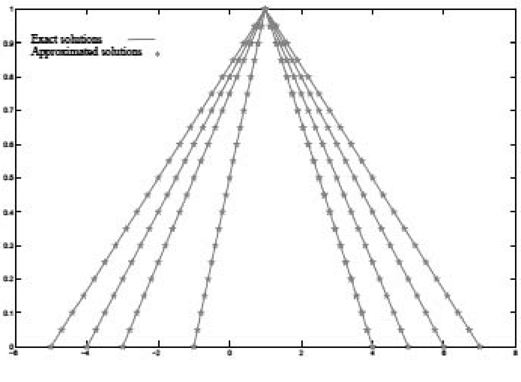

Figure 1: Exact solutions and approximated solutions in Example 4.1

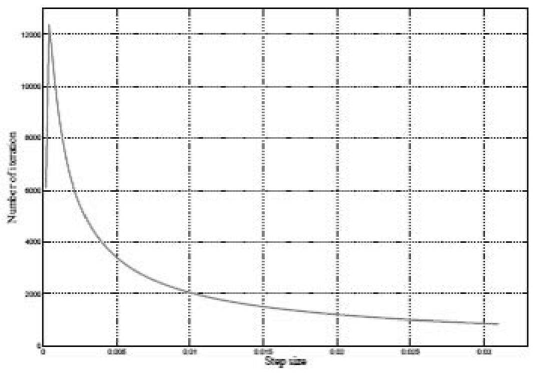

Figure 2: The relation between step size and the number of iterations in Example 4.1

The difference between exact and approximated fuzzy solutions can be seen in Fig. 1. Using solid lines (Exact solution) and the dotted lines (Approximated solutions), respectively, the proximity between the exact triangular fuzzy numbers and their approximation is clearly observed. In addition, the relation between the number of iterations and the step size is presented in Fig. 2. It can be seen that the closer value of the step size is to $2/\|S\|^2$, then the number of iterations needed to obtain the solution decreases.

Example 4.2: One of the applications of Sylvester equation is a special decoupling property of the solution for computing matrix sign function [7]. If $A = Z\mathrm{diag}(J_1, J_2)Z^{-1}$ is a Jordan canonical form of A where J_1 is a $p \times p$ Jordan block, J_2 is a $q \times q$ Jordan block and Z is a nonsingular matrix, then the matrix sign function is defined as follows

$$\mathrm{sign}(A) = Z\begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix}Z^{-1},$$

By considering $T = \begin{pmatrix} A & \tilde{C} \\ 0 & B \end{pmatrix}$ and $Z = \begin{pmatrix} I_m & \tilde{X} \\ 0 & I_n \end{pmatrix}$. Then the matrix sign function gives an expression for the solution of a FSME:

$$\mathrm{sign}(T) = Z\mathrm{sign}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}Z^{-1} = Z\begin{pmatrix} I_m & 0 \\ 0 & I_n \end{pmatrix}Z^{-1} = \begin{pmatrix} I_m & 2\tilde{X} \\ 0 & I_n \end{pmatrix}$$

Now, suppose

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} (-34 + 9r, 9 - 34r) & (-22 + 18r, 18 - 22r) \\ (-14 + 21r, 21 - 14r) & (-18 + 17r, 17 - 18r) \end{pmatrix}.$$

By taking $= 0.0236$ after 222 iterations, the following solution is obtained

$$\tilde{X} = \begin{pmatrix} (-4.0002 + 1.0002r, 1.0002 - 4.0002r) & (-3.0001 + 1.0001r, 1.0001 - 3.0001r) \\ (-1.0001 + 3.0001r, 3.0001 - 1.0001r) & (-1.0000 + 4.0000r, 4.0000 - 1.0000r) \end{pmatrix}.$$

Then sign(T) can be evaluated by using the following expression

$$\mathrm{sign}(T) = \begin{pmatrix} I_2 & 2\tilde{X} \\ 0 & I_2 \end{pmatrix}$$

It should be emphasized that the exact solution of the associated Sylvester equation is

$$\tilde{X} = \begin{pmatrix} (-4 + r, 1 - 4r) & (-3 + r, 1 - 3r) \\ (-1 + 3r, 3 - r) & (-1 + 4r, 4 - r) \end{pmatrix}$$

In Fig. 5 the exact solutions and approximated solutions can be compared. In addition the relation between the number of iteration and step size is presented in Fig. 6. According to the Fig. 6, it is recognized that the number of iterations is again decreasing function when the step size approaches to $\frac{2}{\|S\|^2}$.

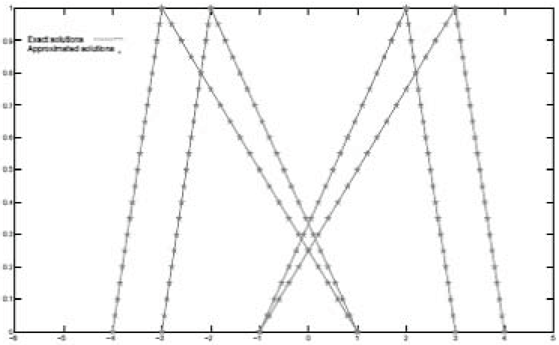

Figure 3: Exact solutions and approximated solutions in Example 4.2

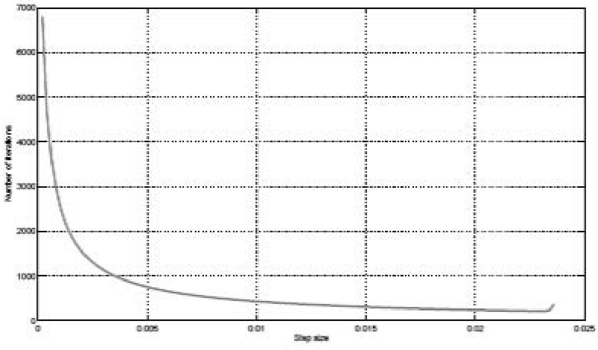

Figure 4: The relation between step size and the number of iterations in Example 4.2

6 Conclusion

In this note, the general analysis for solving the fuzzy Sylvester matrix equation is described. The original fuzzy system utilizing known operators is replaced by a crisp linear system. A numerical iterative method is proposed for solving fuzzy linear system. The important advantage of this iteration that is each n n square system can be solved efficiently instead of $2n$ $2n$ system. As we know, the used computational complexity in this case is less than solving one $2n$ $2n$. Furthermore, the relation between step size parameter and the number of iteration is investigated. It is concluded that if parameter approaches to $\frac{2}{\|S\|^2}$ then the number of iterations is decreased. Examples were given to demonstrate the theory.

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some geometric properties of a new sequence space

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{QT} s³ª·¢´µ¢·¢L pNdN ¢¯¥ k¶®¢³L sNL g¦¯¦³¢-ª»¦¥ ·¦¤µ°³M·¢-¶¦¥ ±¢³¢¯°³®¦¥ ´¦²¶¦¯¤¦ ´±¢¤¦ ¶´ª¯¨ ®°¥¶-¶´ §¶¯¤µª°¯L a±±-N m¢µ©N c°®±¶µN RQU HRPQPI TQQP5TQQXN

{QU} t³ª±¢µ©ºL bN cNL a-µª¯ yN ¢¯¥ eµL mNL g¦¯¦³¢-ª»¦¥ ¥ª;¦³¦¯¤¦ ´¦²¶¦¯¤¦ ´±¢¤¦´ °¯ ´¦®ª¯°³®¦¥ ´±¢¤¦ ¥¦<¯¦¥ £º o³-ª¤» §¶¯¤µª°¯´L m¢µ©N s-°·¢¤¢ UXHSI HRPPXIL SQU5SRTN

{QV} c°-¢¬L rN ¢¯¥ eµL mNL o¯ ´°®¦ ¥ª;¦³¦¯¤¦ ´¦²¶¦¯¤¦ ´¦µ´ ¢¯¥ µ©¦ª³ µ°±°-°¨ª¤¢- ±³°±¦³µª¦´L b¶--N m¢-¢º´N m¢µ©N s¤ªN s°¤N HRI RXHRI HRPPUIL QRU5QSPN

{QW} k©¢¯L vNaNL a¯¦¸µº±¦ °§ ¥ª;¦³¦¯¤¦ ´¦²¶¦¯¤¦ ´±¢¤¦´L a±±-N s¤ªN QR HRPQPIL QPR5QPXN

{QX} c¶ªL yN ¢¯¥ h¶¥»ª¬L hNL o¯ µ©¦ b¢¯¢¤©Ms¢¬´ ¢¯¥ ¸¦¢¬ b¢¯¢¤©Ms¢¬´ ±³°±¦³µª¦´ °§ ´°®¦ b¢¯¢¤© ´¦²¶¦¯¤¦ ´±¢¤¦´L a¤µ¢ s¤ª¦¯µª¢³¶® m¢µ©¦®¢µª¤¢³¶® VUHQMRI HQYYYIL QWY5QXWN

{QY} m¶³´¢-¦¦¯L mNL Ĉ°-¢¬L rN ¢¯¥ eµL mNL s°®¦ ¨¦°®¦µ³ª¤ ª¯¦²¶¢-ªµª¦´ ª¯ ¢¯¦¸ b¢¯¢¤© ´¦²¶¦¯¤¦ ´±¢¤¦L jN i¯¦²¶¢-N a±±-N RPPWL a³µN id XVWUWL V ±±N

{RP} c©ª¥¶®¦L cN eNL g¦°®¦µ³ª¤ p³°±¦³µª¦´ °§ b¢¯¢¤© s±¢¤¦´ ¢¯¥ n°¯M-ª¯¦¢³ iµ¦³¢µª°¯´L l¦¤µ¶³¦ n°µ¦´ ª¯ m¢µ©¦®¢µª¤´L s±³ª¯¨¦³Mv¦³-¢¨L h¦ª¥¦-£¦³¨L b¦³-ª¯L QYVUN

{RQ} g¶³¢³ªªL vN iNL dª;¦³¦¯µª¢- p³°±¦³µª¦´ °§ µ©¦ c°¯·¦¹ªµº m°¥¶-ª °§ b¢¯¢¤© s±¢¤¦´L m¢µ¦®M¢µª¤©¦´¬ª¦ i´´-¦¥°·¢¯ªº¢ RHQI HQYVWIL QTQMQTXN

{RR} s¢¯¤©¦»L lN ¢¯¥ u--¢¯L aNL s°®¦ ±³°±¦³µª¦´ °§ g¶³¢³ªªG´ ®°¥¶-¶´ °§ ¤°¯·¦¹ªµºL a³¤©N m¢µ©N WQHUI HQYYXIL SYY5TPVNL

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Approximation Theorems for Complex Szasz-Kantorovich Operators

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Abstract

In this paper, the order of simultaneous approximation and Voronovskaja-type results with quantitative estimates for complex Szász-Kantorovich operators attached to analytic functions on compact disks are obtained.

Keywords. Szasz–Mirakjan operator, Szasz-Kantorovich operator, complex approximation.

1 Introduction

For a real function of real variable $f:[0,\infty)\to\mathbb{R}$, it is well known that the Favard–Szasz–Mirakjan operators are given by

$$S_n(f;x)=e^{-nx}\sum_{j=0}^{\infty}\frac{(nx)^j}{j!}f\left(\frac{j}{n}\right),$$

where for the convergence of $S_n(f;x)$ to $f(x)$, usually f is supposed to be of exponential growth, that is $|f(x)|\leq C\exp(Bx)$, for all $x\in[0,\infty)$, with $C,B>0$, (see Favard [1]). Also, concerning quantitative estimates in approximation of $f(x)$ by $S_n(f;x)$, in e.g. [2], it is proved that under some additional assumptions on f, we actually have $|S_n(f;x)-f(x)|\leq\frac{C}{n}$, for all $x\in[0,\infty)$, $n\in\mathbb{N}.$ In [4] Gal, under the condition that $f:[0,\infty)\to\mathbb{C}$ of exponential growth, obtained quantitative estimates in closed disks with center in origin. Unlike the convergence results in [5], all the results in the present paper are obtained in the absence of the exponential-type growth conditions for analytic f in the disk. Approximation properties of the q-Szasz-Mirakjan operators are studied in [6].

The goal of the present note is to extend these type of results to complex Szasz-Kantorovich operators defined as follows

$$K_n(f;z)=e^{-nz}\sum_{j=0}^{\infty}\frac{(nz)^j}{j!}\int_0^1 f\left(\frac{j+t}{n+1}\right)dt. \tag{1}$$

If f is bounded on $[0,\infty)$ then it is clear that $K_n(f;z)$ are well defined for all $z\in\mathbb{C}$.

Let $\mathbb{D}_R$ be a disc $\mathbb{D}_R:=\{z\in\mathbb{C}:|z|<R\}$ in the complex plane $\mathbb{C}$. Denote by $H(\mathbb{D}_R)$ the space of all analytic functions on $\mathbb{D}_R$. For $f\in H(\mathbb{D}_R)$ we assume that $f(z)=\sum_{m=0}^{\infty}a_m z^m$.

We start with the following quantitative estimates of the convergence for complex Szasz-Kantorovich operators attached to an analytic function in a disk of radius $R>1$ and center 0.

Theorem 1 *Let $f\in H(\mathbb{D}_R)$ and $f:[R,\infty)\cup\overline{\mathbb{D}_R}\to\mathbb{C}$ be bounded on $[0,\infty)$. If $1\leq r<\frac{R}{2}$, then for all $|z|\leq r$ and $n\in\mathbb{N}$ we have*

$$|K_n(f;z)-f(z)|\leq\frac{3}{2n}\sum_{m=1}^{\infty}|c_m|\,m(m+1)(2r)^m.$$

The next theorem gives a Voronovskaja type result in compact disks, for complex Szasz-Kantorovich operators attached to an analytic function in $\mathbb{D}_R$, $R > 1$ and center 0.

Theorem 2 *Let $f \in H(\mathbb{D}_R)$ and $f : [R, \infty) \cup \overline{\mathbb{D}_R} \to \mathbb{C}$ be bounded on $[0, \infty)$. If $1 \le r < \frac{R}{2}$ then for all $|z| \le r$ and $n \in \mathbb{N}$ we have*

$$\left| K_{n,q}(f;z) - f(z) - \frac{1-2z}{2(n+1)} f'(z) - \frac{z(1-z)}{2(n+1)} f''(z) \right| \le \frac{10}{n^2} \sum_{m=2}^{\infty} |a_m| m (m-1)^3 (2r)^m .$$

As an application of Theorem 2 we present the order of approximation for complex q-Kantorovich operators.

Theorem 3 *Let $f \in H(\mathbb{D}_R)$ and $f : [R, \infty) \cup \overline{\mathbb{D}_R} \to \mathbb{C}$ be bounded on $[0, \infty)$. If $1 \le r < \frac{R}{2}$ and if f is not a constant function then the estimate*

$$\| K_n(f) - f \|_r \ge \frac{1}{n} C_r(f), \quad n \in \mathbb{N},$$

holds, where the constant $C_r(f)$ depends on f and r but it is independent of n.

2 Auxiliary results

Lemma 4 *For all $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $z \in \mathbb{C}$ we have*

$$K_n(e_m;z) = \frac{1}{(n+1)^m} \sum_{k=0}^{m} \binom{m}{k} \frac{n^k}{m-k+1} S_n(e_k;z). \tag{2}$$

Proof. The recurrence formula can be derived by direct computation:

$$\begin{aligned}
K_n(e_m;z) &= e^{-nz} \sum_{j=0}^{\infty} \frac{(nz)^j}{j!} \int_0^1 \left(\frac{j+t}{n+1} \right)^m dt = e^{-nz} \sum_{j=0}^{\infty} \frac{(nz)^j}{j!} \sum_{k=0}^{m} \binom{m}{k} \int_0^1 \frac{j^t t^{m-k}}{(n+1)^m} dt \\
&= e^{-nz} \sum_{j=0}^{\infty} \frac{(nz)^j}{j!} \sum_{k=0}^{m} \binom{m}{k} \frac{j^k}{(n+1)^m (m-k+1)} dt \\
&= \sum_{k=0}^{m} \binom{m}{k} \frac{n^k}{(n+1)^m (m-k+1)} e^{-nz} \sum_{j=0}^{\infty} \frac{(nz)^j}{j!} \left(\frac{j}{n} \right)^k \\
&= \frac{1}{(n+1)^m} \sum_{k=0}^{m} \binom{m}{k} \frac{n^k}{(m-k+1)} S_n(e_k;z).
\end{aligned}$$

■

Lemma 5 *For all $z \in \mathbb{C}$ we have*

$$|K_n(e_m;z)| \le (2r)^m, \quad m \in \mathbb{N}.$$

Proof. Indeed, using the inequality $|S_n(e_j;z)| \le (2r)^j$ from [3] p. 115, we get

$$\begin{aligned}
|K_n(e_m;z)| &\le \sum_{j=0}^{m} \binom{m}{j} \frac{n^j}{(m-j+1)} |S_n(e_j;z)| \\
&\le |S_n(e_m;z)| \le (2r)^m .
\end{aligned}$$

■

Lemma 6 *We have*

$$K_n(e_0;z)=1;\qquad K_n(e_1;z)=\frac{1}{2(n+1)}+\frac{n}{n+1}z;$$
$$K_n(e_2;z)=\frac{1}{3(n+1)^2}+\frac{2n}{(n+1)^2}z+\frac{n^2}{(n+1)^2}z^2;$$
$$K_n\left((e_1-xe_0)^2;z\right)=\frac{1}{3(n+1)^2}+\frac{n-1}{(n+1)^2}z+\frac{1}{(n+1)^2}z^2.$$

Lemma 7 *For all* $n,m\in\mathbb{N}$, $z\in\mathbb{C}$ *we have*

$$\begin{aligned}K_n(e_{m+1};z)&=\frac{z}{n}K_n'(e_m;z)+zK_n(e_m;z)\\&+\frac{1}{(n+1)^m}\sum_{j=0}^{m+1}\binom{m+1}{j}\frac{n^j}{(m-j+2)}\left(\frac{1}{n+1}-\frac{j}{(m+1)n}\right)S_n(e_j;z).\end{aligned}\tag{3}$$

Proof. We know that, (see [3] p. 115)

$$S_n'(e_j;z)=-nS_n(e_j;z)+\frac{n}{z}S_n(e_{j+1};z).$$

Taking the derivative of the formula (2) and using the above formula we have

$$\begin{aligned}K_n'(e_m;z)&=\frac{1}{(n+1)^m}\sum_{j=0}^{m}\binom{m}{j}\frac{n^j}{(m-j+1)}\left(-nS_n(e_j;z)+\frac{n}{z}S_n(e_{j+1};z)\right)\\&=\frac{n}{z}\frac{1}{(n+1)^m}\sum_{j=0}^{m}\binom{m}{j}\frac{n^j}{(m-j+1)}S_n(e_{j+1};z)-n\frac{1}{(n+1)^m}\sum_{j=0}^{m}\binom{m}{j}\frac{n^j}{(m-j+1)}S_n(e_j;z)\end{aligned}$$

$$\frac{z}{n}K_n'(e_m;z)=\frac{1}{(n+1)^m}\sum_{j=1}^{m+1}\binom{m}{j-1}\frac{n^{j-1}}{(m-j+2)}S_n(e_j;z)-zK_n(e_m;z).$$

It follows that

$$\begin{aligned}K_n(e_{m+1};z)&=\frac{z}{n}K_n'(e_m;z)+zK_n(e_m;z)\\&+\frac{1}{(n+1)^{m+1}}\sum_{j=0}^{m+1}\binom{m+1}{j}\frac{n^j}{(m-j+2)}S_n(e_j;z)\\&-\frac{1}{(n+1)^m}\sum_{j=1}^{m+1}\binom{m}{j-1}\frac{n^{j-1}}{(m-j+2)}S_n(e_j;z)\\&=\frac{z}{n}K_n'(e_m;z)+zK_n(e_m;z)+\frac{1}{n^{m+1}(m+2)}\\&+\frac{1}{(n+1)^m}\sum_{j=1}^{m+1}\binom{m+1}{j}\frac{n^j}{(m-j+2)}\left(\frac{1}{n+1}-\frac{j}{(m+1)n}\right)S_n(e_j;z)\\&=\frac{z}{n}K_n'(e_m;z)+zK_n(e_m;z)\\&+\frac{1}{(n+1)^m}\sum_{j=0}^{m+1}\binom{m+1}{j}\frac{n^j}{(m-j+2)}\left(\frac{1}{n+1}-\frac{j}{(m+1)n}\right)S_n(e_j;z).\end{aligned}$$

Here we used the identity

$$\binom{m}{j-1}=\binom{m+1}{j}\frac{j}{(m+1)}.$$

■

Define

$$E_{n,m}(z) := K_n(e_m;z) - e_m(z) - \frac{m^2 - 2mz\, z^{m-1}}{2(n+1)}.$$

Lemma 8 *Let $n,m \in \mathbb{N}$, we have the following recurrence formula*

$$E_{n,m}(z) = \frac{z}{n}\left(K_n(e_{m-1};z) - e_{m-1}(z)\right)' + zE_{n,m-1}(z) + \frac{m-1}{n(n+1)}z^{m-1} - \frac{1}{2(n+1)}z^{m-1} + \frac{1}{n+1}z^m + \frac{1}{(n+1)^m}\sum_{j=0}^{m}\binom{m}{j}\frac{n^j}{(m-j+1)}\left(1-\frac{j}{m}-\frac{j}{mn}\right)S_n(e_j;z).$$

Proof. It is immediate that $E_{n,m}(z)$ is a polynomial of degree less than or equal to m and that $E_{n,0}(z) = 0$.

Using the formula (3) we get

$$\begin{aligned}E_{n,m}(z) &= \frac{z}{n}\left\{\left(K_n(e_{m-1};z) - e_{m-1}(z)\right)' + \left(z^{m-1}\right)'\right\} \\ &\quad + z\left\{E_{n,m-1}(z) + z^{m-1} + \frac{(m-1)^2 - 2(m-1)z\, z^{m-2}}{2(n+1)}\right\} \\ &\quad + \frac{1}{(n+1)^{m-1}}\sum_{j=0}^{m}\binom{m}{j}\frac{n^j}{(m-j+1)}\left(\frac{1}{n+1}-\frac{j}{mn}\right)S_n(e_j;z) - z^m - \frac{m^2 - 2mz\, z^{m-1}}{2(n+1)} \\ &= \frac{z}{n}\left(K_n(e_{m-1};z) - e_{m-1}(z)\right)' + \frac{m-1}{n}z^{m-1} + zE_{n,m-1} + \frac{(m-1)^2 - 2(m-1)z\, z^{m-1}}{2(n+1)} \\ &\quad + \frac{1}{(n+1)^{m-1}}\sum_{j=0}^{m}\binom{m}{j}\frac{n^j}{(m-j+1)}\left(\frac{1}{n+1}-\frac{j}{mn}\right)S_n(e_j;z) - \frac{m^2 - 2mz\, z^{m-1}}{2(n+1)}\end{aligned}$$

which is the desired recurrence formula. ■

3 Proofs of the main results

Proof of Theorem 1. Using the recurrence formula (3) we obtain the following relationship:

$$\begin{aligned}K_{n,q}(e_m;z) - e_m(z) &= \frac{z}{n}K_n'(e_{m-1};z) + z\left(K_n(e_{m-1};z) - e_{m-1}(z)\right) \\ &\quad + \frac{1}{(n+1)^m}\sum_{j=0}^{m}\binom{m}{j}\frac{n^j}{(m-j+1)}\left(1-\frac{j}{m}-\frac{j}{mn}\right)S_n(e_j;z). \qquad (4)\end{aligned}$$

We can easily estimate the sum in the above formula as follows:

$$\begin{aligned}&\frac{1}{(n+1)^m}\sum_{j=0}^{m}\binom{m}{j}\frac{n^j}{(m-j+1)}\left(1-\frac{j}{m}-\frac{j}{mn}\right)S_n(e_j;z) \\ &\le \frac{1}{(n+1)^m}\left\{\sum_{j=0}^{m-1}\binom{m-1}{j}\frac{m}{m-j}\frac{n^j}{m-j+1}\left|1-\frac{j}{m}-\frac{j}{mn}\right|\right\}|S_n(e_j;z)| \\ &\quad + \frac{n^{m-1}}{(n+1)^m}S_n(e_m;z) \\ &\le \frac{2m(n+1)^{m-1} + n^{m-1}}{(n+1)^m}(2r)^m \le \frac{2m+1}{n+1}(2r)^m. \qquad (5)\end{aligned}$$

It is known that by a linear transformation, the Bernstein inequality in the closed unit disk becomes

$$|P_m'(z)| \le \frac{m}{r}\|P_m\|_r, \quad \text{for all } |z| \le r,\ r \ge 1,$$

where $P_m(z)$ is a complex polynomial of degree $\leq m$. From the above recurrence formula (4) we get

$$\begin{aligned}|K_{n;q}(e_m;z) - e_m(z)| &\leq \frac{|z|}{n}\left|K_n'(e_{m-1};z)\right| + |z||K_n(e_{m-1};z) - e_{m-1}(z)| + \frac{(2m+1)}{n+1}(2r)^m \\ &\leq \frac{r}{n}\frac{m-1}{r}\|K_n(e_{m-1})\|_r + r|K_n(e_{m-1};z) - e_{m-1}(z)| + \frac{(2m+1)}{n+1}(2r)^m \\ &\leq r|K_{n;q}(e_{m-1};z) - e_{m-1}(z)| + \frac{3m}{n}(2r)^m.\end{aligned}$$

Writing the last inequality for $k = 1, 2, \ldots,$ we easily obtain,

$$\begin{aligned}|K_n(e_m;z) - e_m(z)| &\leq \frac{(2r)^m}{n}3m + r\frac{(2r)^{m-1}}{n}3(m-1) + r^2\frac{(2r)^{m-2}}{n} + \ldots + r^{m-1}\frac{(2r)}{n}3 \\ &= \frac{3(2r)^m}{n}(m + m - 1 + \ldots + 1) \leq \frac{3m(m+1)}{2n}(2r)^m. \qquad (6)\end{aligned}$$

Since $K_n(f;z)$ is analytic in $\mathbb{D}_R,$ we can write

$$K_n(f;z) = \sum_{m=0}^{\infty} a_m K_n(e_m;z), \quad z \in \mathbb{D}_R,$$

which, together with estimate (6) immediately implies for all $|z| \leq r$

$$|K_n(f;z) - f(z)| \leq \sum_{m=0}^{\infty} |a_m||K_n(e_m;z) - e_m(z)| \leq \frac{3}{2n}\sum_{m=1}^{\infty} |c_m| m(m+1)(2r)^m.$$

■

Proof of Theorem 2. A simple calculation and applying the recurrence formula (3) lead us to the following relationship

$$\begin{aligned}E_{n;m}(z) = {} & \frac{z}{n}\left(K_n(e_{m-1};z) - e_{m-1}(z)\right)' + zE_{n;m-1}(z) + \frac{m-1}{n(n+1)}z^{m-1} \\ & + \frac{1}{n+1}\left(z^m - S_n(e_m;z)\right) + \frac{1}{n+1}\left(1 - \frac{n^{m-1}}{(n+1)^{m-1}}\right)S_n(e_m;z) \\ & - \frac{n^{m-2}}{2(n+1)^m}(m-1)S_n(e_{m-1};z) + \frac{1}{2(n+1)}\left(S_n(e_{m-1};z) - z^{m-1}\right) \\ & - \frac{1}{2(n+1)}\left(1 - \frac{n^{m-1}}{(n+1)^{m-1}}\right)S_n(e_{m-1};z) \\ & + \frac{1}{(n+1)^m}\sum_{j=0}^{m-2}\binom{m}{j}\frac{n^j}{(m-j+1)}\left(1 - \frac{j}{m} - \frac{j}{mn}\right)S_n(e_j;z) \\ := {} & \sum_{k=1}^{9} I_k.\end{aligned}$$

From the proof of Theorem 1.8.4 of [3], we have

$$|z^m - S_n(e_m;z)| \leq \frac{6(m-1)}{n}(2r)^{m-1}.$$

It follows that

$$|I_4| \leq \frac{1}{n+1}|z^m - S_n(e_m;z)| \leq \frac{6(m-1)(2r)^{m-1}}{n(n+1)},$$

$$|I_7| \leq \frac{1}{2(n+1)}\left|z^{m-1} - S_n(e_{m-1};z)\right| \leq \frac{3(m-2)(2r)^{m-2}}{n(n+1)}.$$

Applying the inequality

$$1-\prod_{j=1}^{k} x_j \le \sum_{j=1}^{k}(1-x_j), \quad 0 \le x_j \le 1, \quad j=1,\dots,k$$

we have

$$|I_5| \le \frac{1}{n+1}\left(1-\frac{n^{m-1}}{(n+1)^{m-1}}\right)|S_n(e_m;z)| \le \frac{m-1}{(n+1)^2}(2r)^m$$

$$|I_8| \le \frac{1}{2(n+1)}\left(1-\frac{n^{m-1}}{(n+1)^{m-1}}\right)|S_n(e_{m-1};z)| \le \frac{m-1}{2(n+1)^2}(2r)^{m-1}.$$

For I_9 we have

$$|I_9| \le \frac{1}{(n+1)^m}\sum_{j=0}^{m-2}\binom{m-2}{j}\frac{m(m-1)}{(m-j)(m-j-1)}\frac{n^j}{(m-j+1)}\left(1-\frac{j}{m}-\frac{j}{mn}\right)|S_n(e_j;z)|$$
$$\le \frac{2m(m-1)(n+1)^{m-2}}{(n+1)^m}(2r)^{m-2} \le \frac{2m(m-1)}{(n+1)^2}(2r)^{m-2}.$$

Thus

$$|E_{n,m}(z)| \le \frac{r}{n}\left|(K_n(e_{m-1};z)-e_{m-1}(z))'\right| + r|E_{n,m-1}(z)| + \frac{m-1}{n(n+1)}r^{m-1} + \frac{6(m-1)}{n(n+1)}(2r)^{m-1}$$
$$+\frac{m-1}{(n+1)^2}(2r)^m + \frac{m-1}{2(n+1)^2}(2r)^{m-1} + \frac{3(m-2)(2r)^{m-2}}{n(n+1)}$$
$$+\frac{m-1}{2(n+1)^2}(2r)^{m-1} + \frac{2m(m-1)}{(n+1)^2}(2r)^{m-2}$$
$$\le \frac{r}{n}\frac{m-1}{r}\frac{3m(m-1)}{2n}(2r)^m + r|E_{n,m-1}(z)| + \frac{8m(m-1)}{(n+1)^2}(2r)^m$$
$$\le r|E_{n,m-1}(z)| + \frac{10m(m-1)^2}{n^2}(2r)^m$$

$$|E_{n,m}(z)| \le r|E_{n,m-1}(z)| + \frac{10m(m-1)^2}{n^2}(2r)^m$$

As a consequence, we get

$$|E_{n,m}(z)| \le \frac{10m(m-1)^3}{n^2}(2r)^m.$$

Note that since $f^{(4)}=\sum_{m=4}^{\infty}a_m m(m-1)(m-2)(m-3)z^{m-4}$ and the series is absolutely convergent for all $|z|<R$, it easily follows the finiteness of the involved constants in the statement. ∎

Proof of Theorem 3. For all $z\in\mathbb{D}_R$ and $n\in\mathbb{N}$ we get

$$K_n(f;z)-f(z)=\frac{1}{n}\left(\frac{(1-2z)}{2}f'(z)+\frac{z}{2}f''(z)+n\left(K_n(f;z)-f(z)-\frac{(1-2z)}{2n}f'(z)-\frac{z}{2n}f''(z)\right)\right).$$

We apply

$$\|F+G\|_r \ge \left|\|F\|_r-\|G\|_r\right| \ge \|F\|_r-\|G\|_r$$

to get

$$\|K_n(f)-f\|_r \ge \frac{1}{n}\left(\left\|\frac{(1-2z)}{2}f'(z)+\frac{z}{2}f''(z)\right\|_r - n\left\|K_n(f;z)-f(z)-\frac{(1-2z)}{2n}f'(z)-\frac{z}{2n}f''(z)\right\|_r\right).$$

Taking into account that by hypothesis f is not a polynomial of degree 0 in D_R, we get $\|e_1(1-e_1)f''-e_1f'\|_r>0$.

Indeed, supposing the contrary it follows that $(1-2z)f'(z)+zf''(z)=0$ for all $|z|\le r$, that is $(zf'(z))'-2zf'(z)=0$ for all $|z|\le r$. The last equality is equivalent to $zf'(z)=Ce^{2z}$ for all $|z|\le r$ with $z\neq 0$. Therefore we get $f'(z)=C\frac{e^{2z}}{z}$, for all $|z|\le r$ with $z\neq 0$. But since f is analytic in $\overline{D_r}$, we necessarily have $C=0$, which implies $f'(z)=0$ and $f(z)=c$ for all $z\in\overline{D_r}$, a contradiction with the hypothesis.

Now, by Theorem 2 we have

$$\left|n\left(K_n(f;z)-f(z)-\frac{(1-2z)}{2n}f'(z)-\frac{z}{2n}f''(z)\right)\right|\le\frac{10}{n}\sum_{m=2}^{\infty}|a_m|m(m-1)^3(2r)^m\to 0 \text{ as } n\to\infty.$$

Consequently, there exists n_1 (depending only on f and r) such that for all $n\ge n_1$ we have

$$\begin{aligned}&\left\|\frac{(1-2z)}{2}f'(z)+\frac{z}{2}f''(z)\right\|_r-\left\|n\left(K_n(f;z)-f(z)-\frac{(1-2z)}{2n}f'(z)-\frac{z}{2n}f''(z)\right)\right\|_r\\ &\ge\frac{1}{2}\left\|\frac{(1-2z)}{2}f'(z)+\frac{z}{2}f''(z)\right\|_r,\end{aligned}$$

which implies

$$\|K_n(f)-f\|_r\ge\frac{1}{2n}\left\|\frac{(1-2z)}{2}f'(z)+\frac{z}{2}f''(z)\right\|_r,\quad\text{for all } n\ge n_1.$$

For $1\le n\le n_1-1$ we have

$$\|K_n(f)-f\|_r\ge\frac{1}{n}\left(n\|K_n(f)-f\|_r\right)=\frac{1}{n}M_{r;n}(f)>0,$$

which finally implies that

$$\|K_n(f)-f\|_r\ge\frac{1}{n}C_r(f),$$

for all n, with $C_r(f)=\min\left\{M_{r;1}(f),\dots,M_{r;n_1-1}(f),\frac{1}{2}\left\|\frac{(1-2z)}{2}f'(z)+\frac{z}{2}f''(z)\right\|_r\right\}$. ■

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IDENTITIES ON THE MODIFIED q-EULER AND q-BERNSTEIN POLYNOMIALS AND NUMBERS WITH WEIGHT

SEOG-HOON RIM AND JOOHEE JEONG

Abstract. Recently, the modified q-Euler numbers and polynomials with weight α are introduced in [15]. In this paper, we give some interesting identities on the modified q-Euler numbers and polynomials with weight α and q-Bernstein polynomials. These results are different from those in [11].

1. Introduction

Let p be a fixed odd prime number. Throughout, this paper $\mathbb{Z}_p, \mathbb{Q}_p$, and $\mathbb{C}_p$ will respectively denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \bigcup \{0\}$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = 1/p$. Let $q \in \mathbb{C}$ with $|1-q|_p < 1$. As a notation of q-numbers, $[x]_q$ is defined by $[x]_q = \frac{1-q^x}{1-q}$,(cf. [1–16]) In this paper, we assume that $\alpha \in \mathbb{Q}_p$. Note that $\lim_{q\to 1}[x]_q = x$. Let $C(\mathbb{Z}_p)$ be the space of continuous function on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the q-Bernstein operator is introduced in [2, 3, 6, 11, 15] as follows :

$$
\begin{aligned}
\mathbb{B}_n^{(\alpha)}(f : x) &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}[x]_q^k[1-x]_{\frac{1}{q}}^{n-k} \\
&= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)B_{k,n}^{(\alpha)}(x;q).
\end{aligned} \tag{1}
$$

Here, $\mathbb{B}_n^{(\alpha)}(f : x)$ is called the weighted q-Bernstain operator of order n for f. For $k, n \in \mathbb{Z}_+$, the weighted q-Bernstein polynomial of degree n is defined by

$$
B_{k,n}^{(\alpha)}(x) = \binom{n}{k}[x]_q^k[1-x]_{\frac{1}{q}}^{n-k}, \quad \text{(see [4, 7, 11])}. \tag{2}
$$

For $f \in C(\mathbb{Z}_p)$, the p-adic invariant q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$
\begin{aligned}
I_{-q}(f) &= \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) \\
&= \lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{x=0}^{p^N-1} f(x)\mu_{-q}(x+p^N\mathbb{Z}_p) \\
&= \lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{x=0}^{p^N-1} f(x)(-q)^x; \quad \text{(see [5])}.
\end{aligned} \tag{3}
$$

Let $f_1(x) = f(x+1)$. Then we see that

$$
I_{-q}(q^{-x}f_1) + I_{-q}(q^{-x}f) = [2]_q f(0) \quad \text{(see [5, 6])} \tag{4}
$$

In [15], the modified q-Euler numbers and polynomials with weight α are defined as follows:

$$\begin{aligned}\tilde{E}^{(\alpha)}_{n,q} &= \int_{\mathbb{Z}_p} q^{-x}[x]^n_{q^\alpha}\, d\mu_{-q}(x), \quad \text{for } n\in\mathbb{Z}_+\\ &= \frac{[2]_q}{(1-q^\alpha)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{q^{\alpha l}}{1+q^{\alpha l}} \qquad (5)\\ &= [2]_q\sum_{m=0}^{\infty}(-1)^m[m]^n_{q^\alpha}.\end{aligned}$$

From (5), we can derive the following relations on modified q-Euler polynomials with weight α and modified q-Euler numbers with weight α:

$$\begin{aligned}\tilde{E}^{(\alpha)}_{n,q}(x) &= \sum_{l=0}^{n}\binom{n}{l}(-1)^l[x]^{n-l}_{q^\alpha}q^{\alpha lx}\tilde{E}^{(\alpha)}_{l,q} \qquad (6)\\ &= ([x]_{q^\alpha}+q^{\alpha x}\tilde{E}^{(\alpha)}_q)^n,\end{aligned}$$

with the usual convention about replacing $(\tilde{E}^{(\alpha)}_q)^n$ by $\tilde{E}^{(\alpha)}_{n,q}$. (see [11, 15])

By (5) and (6), we get the following recurrence relation for $\tilde{E}^{(\alpha)}_{n,q}$:

$$\begin{aligned}&\tilde{E}^{(\alpha)}_{0,q} = \frac{[2]_q}{2},\\ &(q^\alpha\tilde{E}^{(\alpha)}_q+1)^n+\tilde{E}^{(\alpha)}_{n,q} = \begin{cases}[2]_q, & \text{if } n=0,\\ 0, & \text{if } n>0,\end{cases}\end{aligned} \qquad (7)$$

with the usual convention about replacing $(\tilde{E}^{(\alpha)}_q)^n$ by $\tilde{E}^{(\alpha)}_{n,q}$.

In this paper, we give some interesting properties between the weighted q-Bernstein polynomial and the modified q-Euler numbers with weight by using p-adic invariant q-integral on $\mathbb{Z}_p$. From these properties, we derive identities on the modified q-Euler polynomials with weight and the weighted q-Bernstein polynomials. These results are different from those in [11].

2. Identities on the modified q-Euler polynomials and Bernstein polynomials

The relation of reflection symmetry of the modified q-Euler polynomials with weight α, we evaluate the following p-adic q-integral on $\mathbb{Z}_p$, we have

$$\tilde{E}^{(\alpha)}_{n,q^{-1}}(1-x) = (-1)^n q^{\alpha n}\tilde{E}^{(\alpha)}_{n,q}(x). \quad \text{(see [15])} \qquad (8)$$

From (8), we have

$$\begin{aligned}
\tilde{E}^{(\alpha)}_{n,q}(2) &= ([2]_q + q^2\tilde{E}^{(\alpha)}_{n,q})^n = (q(q\tilde{E}^{(\alpha)}_{q}+1)+1)^n \\
&= \sum_{l=0}^{n}\binom{n}{l} q^{l}(q\tilde{E}^{(\alpha)}_{q}+1)^l \\
&= \tilde{E}^{(\alpha)}_{0,q} + nq(q\tilde{E}^{(\alpha)}_{q}+1)^1 + \sum_{l=2}^{n}\binom{n}{l} q^{l}(q\tilde{E}^{(\alpha)}_{q}+1)^l \qquad (9)\\
&= \frac{[2]_q}{2} + nq\left(\frac{[2]_q}{2} + \tilde{E}^{(\alpha)}_{1,q}\right) + \sum_{l=2}^{n}\binom{n}{l} q^{l}\tilde{E}^{(\alpha)}_{l,q} \\
&= nq\frac{[2]_q}{2} + \sum_{l=0}^{n}\binom{n}{l} q^{l}\tilde{E}^{(\alpha)}_{l,q}, \qquad \text{for } n>0.
\end{aligned}$$

Let $n \in \mathbb{N}$ with $n \geq 2$. Then, by (7) and (9), we obtain the following:

For $n \in \mathbb{N}$ with $n \geq 2$, we have

$$\tilde{E}^{(\alpha)}_{n,q}(2) - nq\frac{[2]_q}{2} = (q\tilde{E}^{(\alpha)}_{q}+1)^n = \tilde{E}^{(\alpha)}_{n,q}. \tag{10}$$

From (2), we have

$$\begin{aligned}
\int_{\mathbb{Z}_p} q^{-x}\tilde{B}^{(\alpha)}_{k,n}(x,q)\,d\mu_{-q}(x) &= \binom{n}{k}\int_{\mathbb{Z}_p} q^{-x}[x]_q^{k}[1-x]_q^{n-k}\,d\mu_{-q}(x) \\
&= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\int_{\mathbb{Z}_p} q^{-x}[x]_q^{k+l}\,d\mu_{-q}(x) \qquad (11)\\
&= \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\tilde{E}^{(\alpha)}_{k+l,q}.
\end{aligned}$$

Thus by (11), we have a proposition:

Proposition 1. For $n, k \in \mathbb{Z}_+$, we have

$$\int_{\mathbb{Z}_p} q^{-x}\tilde{B}^{(\alpha)}_{k,n}(x,q)\,d\mu_{-q}(x) = \binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\tilde{E}^{(\alpha)}_{k+l,q}.$$

By the definition of the weighted q-Bernstein polynomials in (2), we see that

$$\tilde{B}^{(\alpha)}_{k,n}(x,q) = \tilde{B}^{(\alpha)}_{n-k,n}(1-x, 1/q), \tag{12}$$

where $k, n \in \mathbb{Z}_+$ and $x \in \mathbb{Z}_+$. From (12), we have

$$\begin{aligned}
\int_{\mathbb{Z}_p} q^{-x}\tilde{B}^{(\alpha)}_{k,n}(x,q)\,d\mu_{-q}(x) &= \int_{\mathbb{Z}_p} q^{-x}\tilde{B}^{(\alpha)}_{n-k,n}(1-x,1/q)\,d\mu_{-q}(x) \\
&= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\int_{\mathbb{Z}_p} q^{-x}[1-x]_q^{n-l}\,d\mu_{-q}(x) \qquad (13)\\
&= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\tilde{E}^{(\alpha)}_{n-l,q^{-1}}(2).
\end{aligned}$$

Therefore, by (13), we obtain the following theorem :

Theorem 2. For $n,k \in \mathbb{Z}_+$, we have

$$\int_{\mathbb{Z}_p} q^{-x}\widetilde{B}^{(\ )}_{k,n}(x;q)\,d\mu_{-q}(x) = \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\widetilde{E}^{(\ )}_{n-l;q^{-1}}(2).$$

By comparing the coefficients on the both sides of proposition and Theorem 2, we obtain the following corollary :

Corollary 3. For $n,k \in \mathbb{Z}_+$, we have

$$\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^{k+l}\widetilde{E}^{(\ )}_{k+l;q} = \sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\widetilde{E}^{(\ )}_{n-l;q^{-1}}(2).$$

From (10) and Corollary 3, we obtain the following corollary :

Corollary 4. For $n,k \in \mathbb{Z}_+$ with $n > k$, we have

$$\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^{l}\widetilde{E}^{(\ )}_{k+l;q} = \sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\widetilde{E}^{(\ )}_{n-l;q^{-1}}(2).$$

In particular, if $k = 0$, we have

$$\sum_{l=0}^{n}\binom{n}{l}(-1)^{l}\widetilde{E}^{(\ )}_{l;q} = \widetilde{E}^{(\ )}_{n;q^{-1}}(2) = \widetilde{E}^{(\ )}_{n;q} + nq\frac{[2]_q}{2}.$$

For $m,n,k \in \mathbb{Z}_+$ with $m,n > k$. Then we can derive the following equation from the p-adic invariant q-integral on $\mathbb{Z}_p$:

$$
\begin{aligned}
(14)\quad &\int_{\mathbb{Z}_p} q^{-x}\widetilde{B}^{(\ )}_{k,n}(x;q)\widetilde{B}^{(\ )}_{k,m}(x;q)\,d\mu_{-q}(x)\\
&= \binom{n}{k}\binom{m}{k}\int_{\mathbb{Z}_p} q^{-x}[x]_q^{2k}[1-x]_q^{n+m-2k}\,d\mu_{-q}(x)\\
&= \binom{n}{k}\binom{m}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\int_{\mathbb{Z}_p} q^{-x}[1-x]_q^{n+m-l}\,d\mu_{-q}(x)\\
&= \binom{n}{k}\binom{m}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\widetilde{E}^{(\ )}_{n+m-l;q^{-1}}(2).
\end{aligned}
$$

Therefore, by (14), we obtain the following theorem :

Theorem 5. For $k,n,m \in \mathbb{Z}_+,$ with $m,n > k$, we obtain

$$\int_{\mathbb{Z}_p} q^{-x}\widetilde{B}^{(\ )}_{k,n}(x;q)\widetilde{B}^{(\ )}_{k,m}(x;q)\,d\mu_{-q}(x) = \binom{n}{k}\binom{m}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\widetilde{E}^{(\ )}_{n+m-l;q^{-1}}(2).$$

From (14), we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^{-x}\tilde{B}^{(\alpha)}_{k,n}(x;q)\tilde{B}^{(\alpha)}_{k,m}(x;q)\,d\mu_{-q}(x)\\
(15)\qquad &= \binom{n}{k}\binom{m}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\left(2+\tilde{E}^{(\alpha)}_{n+m-l,q^{-1}}\right)\\
&= \binom{n}{k}\binom{m}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\tilde{E}^{(\alpha)}_{n+m-l,q^{-1}}.
\end{aligned}$$

Let $k,n,m\in\mathbb{Z}_+$. By (3)and (4), we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^{-x}\tilde{B}^{(\alpha)}_{k,n}(x;q)\tilde{B}^{(\alpha)}_{k,m}(x;q)\,d\mu_{-q}(x)\\
(16)\qquad &= \binom{n}{k}\binom{m}{k}\int_{\mathbb{Z}_p} q^{-x}[x]_q^{2k}[1-x]_q^{n+m-2k}\,d\mu_{-q}(x)\\
&= \binom{n}{k}\binom{m}{k}\sum_{l=0}^{n+m-2k}\binom{n+m-2k}{l}(-1)^{l}\tilde{E}^{(\alpha)}_{l+2k,q}.
\end{aligned}$$

Therefore, by (16), we obtain the following theorem :

Theorem 6. For $k,n,m\in\mathbb{Z}_+$ with $m,n>k$, we obtain

$$\int_{\mathbb{Z}_p} q^{-x}\tilde{B}^{(\alpha)}_{k,n}(x;q)\tilde{B}^{(\alpha)}_{k,m}(x;q)\,d\mu_{-q}(x) = \binom{n}{k}\binom{m}{k}\sum_{l=0}^{n+m-2k}\binom{n+m-2k}{l}(-1)^{l}\tilde{E}^{(\alpha)}_{l+2k,q}.$$

By comparing the coefficients on the both sides of proposition and Theorem 6 and (16), we obtain the following corollary :

Corollary 7. For $m,n,k\in\mathbb{Z}_+$ with $m,n>k$, we have

$$\sum_{l=0}^{n+m-2k}\binom{n+m-2k}{l}(-1)\tilde{E}^{(\alpha)}_{l+2k,q} = \sum_{l=0}^{2k}\binom{2k}{l}(-1)^{2k+l}\tilde{E}^{(\alpha)}_{n+m-l,q^{-1}},\quad if\quad k>0.$$

In particular, if $k=0$, we have

$$\sum_{l=0}^{n+m}\binom{n+m}{l}(-1)^{l}\tilde{E}^{(\alpha)}_{l,q} = (n+m)q-\frac{[2]_q}{2}+\tilde{E}^{(\alpha)}_{n+m,q^{-1}}.$$

Continuing this process, we obtain the following theorem :

Theorem 8. For $k,n_1,n_2,\ldots,n_s\in\mathbb{Z}_+$ with $n_1+n_2+\ldots+n_s>sk$, we obtain

$$\begin{aligned}
&\sum_{l=0}^{n_1+n_2+\ldots+n_s-sk}\binom{n_1+n_2+\ldots+n_s-sk}{l}(-1)^{l}\tilde{E}^{(\alpha)}_{l+sk,q}\\
&= \sum_{l=0}^{sk}\binom{sk}{l}(-1)^{sk+l}\tilde{E}^{(\alpha)}_{n_1+n_2+\ldots+n_s-l,q^{-1}},\ \text{if } k>0.
\end{aligned}$$

In particular, if $k = 0$, we have

$$\sum_{l=0}^{n_1+n_2+\cdots+n_s}\binom{n_1+n_2+\cdots+n_s}{l}(-1)^l\tilde{E}^{(\omega)}_{l,q} = (n_1+\cdots+n_s)q\left(\frac{[2]_q}{2}+\tilde{E}^{(\omega)}_{n_1+n_2+\cdots+n_s,q^{-1}}\right).$$

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GENERALIZED TERNARY BI-DERIVATIONS ON TERNARY BANACH ALGEBRAS: A FIXED POINT APPROACH

MADJID ESHAGHI GORDJI, GWANG HUI KIM, JUNG RYE LEE*, AND CHOONKIL PARK

Abstract. Using fixed point method, we investigate the Hyers-Ulam stability of generalized bi-derivations on ternary Banach algebras.

1. Introduction and preliminaries

The stability problem of functional equations had been first raised by Ulam (cf. [35]): Let $(G_1, .)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(., .)$. Given $\epsilon > 0$, dose there exist a $\delta > 0$, such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x.y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?
Ulam [34] discusses:
The notion of stability of mathematical theorems considered from a rather general point of view: When is it true that by changing *a little* the hypothesis of a theorem one can still assert that the thesis of the theorem remains true or *approximately* true?
In the other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [15] gave a first affirmative answer to the question of Ulam for Banach spaces.

In 1978, Th.M. Rassias [28] introduced a new functional inequality that we call Cauchy-Rassias inequality and succeeded to extend the result of Hyers by weakening the condition for the Cauchy difference to be unbounded: if there exist an $\epsilon \geq 0$ and $0 \leq p < 1$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E_1$, then there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{|2 - 2^p|}\|x\|^p$$

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for all $x \in E_1$ (see [16, 17, 29, 30, 31]). In 1991, Gajda [13] solved the problem for $1 < p$, which was raised by Th.M. Rassias. In fact, the result of Th.M. Rassias is valid for $1 < p$; moreover, Gajda gave an example that a similar stability result does not hold for $p = 1$. Another example was given by Th.M. Rassias and P. Semrl [32].

In 1994, a generalization of the Th.M. Rassias' theorem was obtained by Gavruta as follows [14].

Suppose $(G,+)$ is an abelian group, E is a Banach space, and that the so-called admissible control function $\varphi : G \times G \to \mathbb{R}$ satisfies

$$\tilde{\varphi}(x,y) := 2^{-1}\sum_{n=0}^{\infty} 2^{-n}\varphi(2^n x, 2^n y) < \infty$$

for all $x, y \in G$. If $f : G \to E$ is a mapping with

$$\|f(x+y) - f(x) - f(y)\| \le \varphi(x,y)$$

for all $x, y \in G$, then there exists a unique mapping $T : G \to E$ such that $T(x+y) = T(x) + T(y)$ and $\|f(x) - T(x)\| \le \tilde{\varphi}(x,x)$ for all $x, y \in G$.

Ternary algebraic operations were considered in the 19-th century by several mathematicians such as A. Cayley [3] who introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii in 1990 ([19]). The comments on physical applications of ternary structures can be found in [1, 20, 21, 33, 36].

A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product $(x,y,z) \mapsto [xyz]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[xy[zvw]] = [x[wzy]v] = [[xyz]wv]$, and satisfies $\|[xyz]\| \le \|x\|.\|y\|.\|z\|$ and $\|[xxx]\| = \|x\|^3$ (see [3], [6]–[12] and [24]).

A $\mathbb{C}$- bilinear mapping $\delta : A \times A \to A$ is called a ternary bi-derivation if it satisfies

$$\delta([abc],d) = [\delta(a,d)bc] + [a\delta(b,d)c] + [ab\delta(c,d)],$$

$$\delta(a,[bcd]) = [\delta(a,b)cd] + [b\delta(a,c)d] + [bc\delta(a,d)]$$

for all $a, b, c, d \in A$ (see [2]). A $\mathbb{C}$-bilinear mapping $D : A \times A \to A$ is called a ternary generalized bi-derivation if there exists a bi-derivation $\delta : A \times A \to A$ such that

$$D([abc],d) = [D(a,d)bc] + [a\delta(b,d)c] + [ab\delta(c,d)],$$

$$D(a,[bcd]) = [D(a,b)cd] + [b\delta(a,c)d] + [bc\delta(a,d)]$$

for all $a, b, c, d \in A$.

Let X and Y be be real or complex linear spaces. For a mapping $f : X \times X \to Y$, consider the functional equation

$$f(a+b, c-d) + f(a-b, c+d) = 2f(a,c) - 2f(b,d). \qquad (1.1)$$

Bae and W. Park [2] investigated the Hyers-Ulam stability of ternary bi-derivations for bi-additive mappings satisfying (1.1).

We recall a fundamental result in fixed point theory.

Theorem 1.1. ([5]) *Suppose that a complete generalized metric space* (X, d) *and a strictly contractive mapping* $J : X \to X$ *with Lipschits constant* $0 < L < 1$ *are given. Then, for a given element* $x \in X$*, exactly one of the following assertions is true:*
either
(1) $d(J^n x, J^{n+1}x) = \infty$ *for all* $n \geq 0$ *or*
(2) *there exists* n_0 *such that* $d(J^n x, J^{n+1}x) < \infty$ *for all* $n \geq n_0$.
Actually, if (a_2) *holds, then the sequence* $J^n x$ *is a convergent to a fixed point* x^* *of* J *and*
(3) x^* *is the unique fixed point of* J *in* $\Lambda := \{y \in X, d(J^{n_0}x, y) < \infty\}$;
(4) $d(y, x^*) \leq \frac{d(y, Jy)}{1-L}$ *for all* $y \in \Lambda$.

In this paper, we investigate the Hyers-Ulam stability of ternary generalized bi-derivations associated with the functional equation

$$f(x - y, t) + f(x, t - s) = 2f(x, t) - f(y, t) - f(x, s).$$

2. Main results

From now on, we assume that A is a ternary Banach algebra. For a given mapping $f : A \times A \to A$, we define the difference operator $E_{\mu,\nu} f : A^4 \to A$ by

$$E_{\mu,\nu} f(a, b, c, d) = f(\mu a - \mu b, \nu c) + f(\mu a, \nu c - \nu d) - \mu\nu(2f(a, c) - f(b, c) - f(a, d))$$

for all $\mu, \nu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C}, |\lambda| = 1\}$ and all $a, b, c, d \in A$.

We establish the stability of ternary generalized bi-derivations.

Theorem 2.1. *Let* $f, g : A \times A \to A$ *be mappings such that* $g(0, 0) = f(0, 0) = 0$. *Let* $\varphi : A^4 \to [0, \infty)$ *be a function such that*

$$\max\{\|E_{\mu,\nu} f(a, b, c, d)\|, \|E_{\mu,\nu} g(a, b, c, d)\|\} \leq \varphi(a, b, c, d), \tag{2.1}$$

$$\begin{aligned} &\max\{\|f([abc], d) - [f(a, d)bc] - [af(b, d)c] - [abf(c, d)]\|,\\ &\|f(a, [bcd]) - [f(a, b)cd] - [bf(a, c)d] - [bcf(a, d)]\|\}\\ &\leq \varphi(a, b, c, d), \end{aligned} \tag{2.2}$$

$$\begin{aligned} &\max\{\|g([abc], d) - [g(a, d)bc] - [af(b, d)c] - [abf(c, d)]\|,\\ &\|g(a, [bcd]) - [g(a, b)cd] - [bf(a, c)d] - [bcf(a, d)]\|\}\\ &\leq \varphi(a, b, c, d), \end{aligned} \tag{2.3}$$

$$\lim_{n \to \infty} \frac{1}{4^n} \varphi(2^n a, 2^n b, 2^n c, 2^n d) = 0$$

for all $;\quad 2\ \mathrm{T}^1$ *and all* $a;b;c;d\ 2\ A:$ *If there exists an* $L < 1$ *such that* $\quad (a;b)$ $4L\quad (\frac{a}{2};\frac{b}{2})$ *for all* $a;b\ 2\ A;$ *where*

$$(a;b) := \ '(0;a;2b;0) + \ '(a;\quad a;2b;b) + \ '(0;0;2b;0) + 3(\ '(a;0;b;\quad b)$$

$$+ \ '(a;0;0;b) + \ '(a;0;0;0) + \ '(0;0;b;0));$$

then there is a unique ternary bi-derivation $\quad : A\quad A\ !\ A$ *and a unique ternary generalized bi-derivation* $D : A\quad A\ !\ A$ *(related to* $\quad$*) such that*

$$maxfkg(a;c)\quad D(a;c)k;kf(a;c)\quad (a;c)kg\quad \frac{L}{1\quad L}\quad (\frac{a}{2};\frac{c}{2})$$

for all $a;c\ 2\ A:$

Proof. It follows from (2.1) that

$$kE_{\ ;\ } f(a;b;c;d)k\quad '(a;b;c;d); \tag{2.4}$$

Setting $\quad = \quad = 1$ in (2.4), we have

$$jjf(a\quad b;c) + f(a;c\quad d)\quad 2f(a;c) + f(b;c) + f(a;d)jj\quad '(a;b;c;d); \tag{2.5}$$

$$jjg(a\quad b;c) + g(a;c\quad d)\quad 2g(a;c) + g(b;c) + g(a;d)jj\quad '(a;b;c;d) \tag{2.6}$$

for all $a;b;c;d\ 2\ A:$ Letting $a = b = d = 0$ in (2.5), shows that

$$jjf(0;c)jj\quad '(0;0;c;0):$$

Letting $b = c = d = 0$ in (2.5), we get

$$kf(a;0)k\quad '(a;0;0;0):$$

Setting $b = \quad a; c = 2d$ in (2.5), we get

$$jjf(2a;2d) + 2f(a;d)\quad 2f(a;2d) + f(\quad a;2d)jj\quad '(a;\quad a;2d;d): \tag{2.7}$$

Putting $a = d = 0$ in (2.5), we conclude that

$$jjf(\quad b;c) + f(b;c)jj\quad '(0;b;c;0) + \ '(0;0;c;0): \tag{2.8}$$

Letting $b = a; c = 2d$ in (2.8), we get

$$jjf(\quad a;2d) + f(a;2d)jj\quad '(0;a;2d;0) + \ '(0;0;2d;0): \tag{2.9}$$

By (2.7) and (2.9), we obtain

$$jjf(2a;2d) + 2f(a;d)\quad 3f(a;2d)jj\quad '(0;a;2d;0) + \ '(a;\quad a;2d;d) + \ '(0;0;2d;0): \tag{2.10}$$

Setting $b = 0$ and $d = \quad c$ in (2.5), we get

$$jjf(a;2c)\quad f(a;c) + f(0;c) + f(a;\quad c)jj\quad '(a;0;c;\quad c): \tag{2.11}$$

Letting $b = c = 0$ in (2.5), we have

$$kf(a;d) + f(a;\quad d)k\quad '(a;0;0;d) + \ '(a;0;0;0): \tag{2.12}$$

Substituting d by c in (2.12) shows that

$$\|f(a,-c)+f(a,c)\| \le \varphi(a,0,0,c)+\varphi(a,0,0,0). \tag{2.13}$$

By (2.11) and (2.13), we obtain

$$\|f(a,2c)-2f(a,c)\| \le \varphi(a,0,c,-c)+\varphi(a,0,0,c)+\varphi(a,0,0,0)+\varphi(0,0,c,0). \tag{2.14}$$

Letting $c=d$ in (2.14) and multiplying both sides by 3, we get

$$\|3f(a,2d)-6f(a,d)\| \le 3(\varphi(a,0,d,-d)+\varphi(a,0,0,d)+\varphi(a,0,0,0)+\varphi(0,0,d,0)). \tag{2.15}$$

By (2.15) and (2.10), we obtain

$$\|f(2a,2d)-4f(a,d)\| \le \psi(a,d).$$

Hence

$$\|\frac{1}{4}f(2a,2d)-f(a,d)\| \le \frac{1}{4}\psi(a,d) \le L\psi(\frac{a}{2},\frac{d}{2}) \tag{2.16}$$

for all $a,d \in A.$

Consider the set $X := \{g \mid g : A \times A \to A, g(0)=0\}$ and introduce the generalized metric on X

$$d(g,h) := \inf\{t \in \mathbb{R}^+ : \|g(a,d)-h(a,d)\| \le t\psi(\frac{a}{2},\frac{d}{2}),\ \forall a,d \in A\}.$$

It is easy to show that (X,d) is a complete generalized metric space.

Now we consider the linear mapping $J : X \to X$ such that

$$J(g)(a,d) := \frac{1}{4}g(2a,2d)$$

for all $g \in X, a,d \in A.$

Let $g,h \in X$ be given such that $d(g,h)=\epsilon.$ Then

$$\|g(a,d)-h(a,d)\| \le \epsilon\psi(\frac{a}{2},\frac{d}{2})$$

for all $a,d \in X.$ Hence

$$\|Jg(a,d)-Jh(a,d)\| = \frac{1}{4}\|g(2a,2d)-h(2a,2d)\| \le \frac{1}{4}\epsilon\psi(a,d) \le L\epsilon$$

for all $a,d \in A.$ So $d(g,h)=\epsilon$ implies that $d(J(g),J(h)) \le L\epsilon.$ This means that

$$d(Jg,Jh) \le Ld(g,h)$$

for all $g,h \in X.$ It follows from (2.16) that $d(f,Jf) \le L.$

By alternative fixed point, J has a unique fixed point in the set $X_1 := \{h \in X : d(f,h) < \infty\}.$ Let δ be the fixed point of $J,$ that is,

$$\delta(2a,2d) = 4\delta(a,d)$$

for all $a,d \in A$, and there exists a $t \in (0,\infty)$ such that

$$\|\delta(a,d)-f(a,d)\| \le t\psi(\frac{a}{2},\frac{d}{2})$$

for all $a,d\in A.$ On the other hand, we have

$$\lim_{n\to\infty} d(J^n f,\psi)=0.$$

It follows that

$$\lim_{n\to\infty}\frac{1}{4^n}f(2^na,2^nd)=\psi(a,d) \tag{2.17}$$

for all $a,d\in A.$ It follows from $d(f,\psi)\le\frac{1}{1-L}d(f,Jf)$ that

$$d(f,\psi)\le\frac{L}{1-L}.$$

This implies that

$$\|f(a,c)-\psi(a,c)\|\le\frac{L}{1-L}\varphi\left(\frac{a}{2},\frac{c}{2}\right) \tag{2.18}$$

for all $a,c\in A.$ It follows from (2.4) and (2.17) that

$$\begin{aligned}
&\|\psi(x-y,t)+\psi(x,t-s)-2\psi(x,t)+\psi(y,t)+\psi(x,s)\|\\
&\quad=\lim_{n\to\infty}\frac{1}{4^n}\|f(2^n(x-y),2^nt)\\
&\quad+f(2^nx,2^n(t-s))-2f(2^nx,2^nt)+f(2^ny,2^nt)+f(2^nx,2^ns)\\
&\quad\le\lim_{n\to\infty}\frac{1}{4^n}\varphi(2^nx,2^ny,2^nt,2^ns)\\
&\quad=0
\end{aligned}$$

for all $x,y,t,s\in A.$ So

$$\psi(x-y,t)+\psi(x,t-s)=2\psi(x,t)-\psi(y,t)-\psi(x,s)$$

for all $x,y,t,s\in A.$ By [12, Lemma 2.2], the mapping $\psi:A\times A\to A$ is $\mathbb{C}$-bilinear.
By (2.6), we can use the same method as above to show that the limit

$$D(a,d)=\lim_{n\to\infty}\frac{1}{4^n}g(2^na,2^nd)$$

exists for all $a,d\in A.$ Moreover, we can show that $D:A\times A\to A$ is $\mathbb{C}$-bilinear mapping satisfying

$$\|g(a,c)-D(a,c)\|\le\frac{L}{1-L}\varphi\left(\frac{a}{2},\frac{c}{2}\right)$$

for all $a,c\in A.$ It follows from (2.2) that

$$\begin{aligned}
&\|\psi([abc],d)-[\psi(a,d)bc]-[a\psi(b,d)c]-[ab\psi(c,d)]\|\\
&=\lim_{n\to\infty}\frac{1}{4^n}(\|f([2^na2^nb2^nc],2^nd)-[f(2^na,2^nd)2^nb2^nc]\\
&\quad-[2^naf(2^nb,2^nd)2^nc]-[2^na2^nbf(2^nc,2^nd)]\|)\\
&\quad\le\lim_{n\to\infty}\frac{1}{4^n}\varphi(2^na,2^nb,2^nc,2^nd)=0
\end{aligned}$$